A STEENROD-MILNOR ACTION ORDERING ON DICKSON INVARIANTS

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ABSTRACT. Let $f : (E(x_1,...,x_k) \otimes P[y_1,...,y_k])^{GL_k} \to (E(x_1,...,x_k) \otimes P[y_1,...,y_k])^{GL_k}$ be a degree preserving Steenrod module map such that f is an isomorphism on degree $2p^{k-1}(p-1)$. Using a particular ordering depending on the dual Milnor basis we show that f is an upper triangular map, hence an isomorphism.

1. Introduction

Motivated by topological questions regarding the cohomology of an infinite (finite) loop space and influenced by the work of Campbell, Cohen, Peterson and Selick in [1] and [2] we study the problem under which conditions is a Steenrod module map between the full rings of invariants of $GL(k, \mathbb{Z}/p\mathbb{Z})$ an isomorphism. In a sequel we study the same problem between certain quotients of the full ring of invariants [4]. It turns out that although the same result holds its proof is more technical.

It is known that given a monomial d^n there exists a unique p-th power Steenrod operation P^{p^m} of smallest degree such that $P^{p^m} d^n \neq 0$. Thus there exists a set consisting of p-th powers of generators $d_{k,i}^{p^{t_i}}$ such that $d_{k,i}^{p^{t_i}} \backslash d^n$ and $t_i + i - 1 = m$. It is obvious that $P^{p^{t_i}}...P^{p^m} d^n \neq 0$. We are interested in finding the longest such sequence of Steenrod operations. Of course it depends on m and i. The required sequence shares the property that $P^{p^{t_{i(l)}}}...P^{p^m} d^n$ is also a monomial according to proposition 1 e). We call such a sequence a **Steenrod-Milnor** action on d^n. Now we iterate this procedure on the monomial $P^{p^{t_i}}...P^{p^m} d^n$ until the resulting monomial is $d_{k,0}^{p^q}$ for the smallest p^q.

Theorem 5 *There exists a sequence of Steenrod-Milnor operations P^Γ such that $P^\Gamma d^n = \lambda d_{n,0}^{p^{l(n)}}$. Here $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$.*

Next, given two monomials d^n and $d^{n'}$ we define an ordering according to their first different Steenrod-Milnor actions $P^{p^{t_{i(l)}}}...P^{p^m}$ and $P^{p^{t'_{i(l')}}}...P^{p^{m'}}$. We call this action a Steenrod-Milnor action ordering. Using this action we prove the following Theorem:

Theorem 6 *Let $f : (E(x_1,...,x_k) \otimes P[y_1,...,y_k])^{GL_k} \to (E(x_1,...,x_k) \otimes P[y_1,...,y_k])^{GL_k}$ be a Steenrod module map which preserves the degree such that $f(d_{k,k-1}) = \lambda d_{k,k-1}$ for $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$. Then f is a lower triangular map with respect to S-M ordering and hence an isomorphism.*

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A consequence of this result is the well known Theorem of Campbell, Peterson and Selick:

Theorem [1]*Let* $f : \Omega_0^\infty S^\infty \to \Omega_0^\infty S^\infty$ be an H-map which induces an isomorphism on $H_{2p-3}(\Omega_0^\infty S^\infty; \mathbb{Z}/p\mathbb{Z})$. If $p > 2$ suppose in addition that f is a loop map or that

$$f_*(d_{2,0})^* = \lambda(d_{2,0})^*$$

for some $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$. Then $f_{(p)}$ is a homotopy equivalence. *Here* $(d_{2,0})^*$ is the hom-dual of the top degree Dickson generator in D_2.

2. A Steenrod-Milnor action ordering on Dickson invariants

We shall recall some well known Theorems concerning the action of the Steenrod algebra on Dickson algebra generators. Let us also recall the full ring of invariants of $GL(k, \mathbb{Z}/p\mathbb{Z})$.

Let E_k stand for $E(x_1, ..., x_k)$ and S_k for $P[y_1, ..., y_k]$. Here $|x_i| = 1$ and $|y_i| = 2$ with $\beta x_i = y_i$.

Theorem 1. *The Dickson algebra* ${S_k}^{GL_k}$ *is a polynomial algebra on* $\{d_{k,0}, ..., d_{k,k-1}\}$.

The Dickson algebra generators are defined bellow.

Theorem 2. [5] *The algebra* $(E_k \otimes S_k)^{GL_k}$ *is a tensor product between the polynomial algebra* D_k *and the* $\mathbb{Z}/p\mathbb{Z}$*-module spanned by the set of elements consisting of the following monomials:*

$$M_{k;s_1,...,s_l} L_k^{p-2};\ 0 \le l \le k-1,\ \textit{and}\ 0 \le s_1 < \cdots < s_l \le k-1.$$

Here $l = 0$ *implies that* $M_k = x_1 ... x_k$. *Its algebra structure is determined by the following relations:*
a) $(M_{k;s_1,...,s_l} L_k^{p-2})^2 = 0$ *for* $0 \le l \le k-1$, *and* $0 \le s_1 < \cdots < s_l \le k-1$.
b) $M_{k;s_1,...,s_l} L_k^{(p-2)} d_{k,k-1}^{m-1} = (-1)^{(k-l)(k-l-1)/2} \prod_{t=1}^{k-l} M_{k;0,...,\widehat{k-s_t},...,k-1} L_k^{p-2}$.
Here $0 \le l \le k-1$, *and* $0 \le s_1 < \cdots < s_l \le k-1$.

The elements above have been defined by Mui in [5] as follows:

$$M_{k;s_1,...,s_l} = \frac{1}{(k-l)!} \begin{vmatrix} x_1 & \cdots & x_1 \\ \vdots & & \vdots \\ x_1 & \cdots & x_k \\ y_1^{p^{s_1}} & \cdots & y_k^{p^{s_1}} \\ \vdots & & \vdots \\ y_1^{p^{s_l}} & \cdots & y_k^{p^{s_l}} \end{vmatrix} \quad d_{k,i} = \frac{L_{k,i}}{L_k}\ L_{k,i} = \begin{vmatrix} y_1 & \cdots & x_k \\ y_1^p & \cdots & y_k^p \\ \vdots & & \vdots \\ y_1^{p^k} & \cdots & y_k^{p^k} \end{vmatrix}$$

Here there are $k-l$ rows of x_i's and the s_i-th's powers are completing the rest of the first determinant, where $0 \le s_1 < \cdots < s_l \le k-1$. The row $y_1^{p^i} ... y_k^{p^i}$ is omitted in the second determinant. $L_k := L_{k,k}$.

$|M_{k;s_1,...,s_l}| = k - l + 2(p^{s_1} + \cdots + p^{s_l})$ and $|L_{k,i}| = 2(1 + \cdots + p^k - p^i)$.

Theorem 3. [3] *1)* $P^{p^j}(d_{k,i}) = \begin{cases} d_{k,i-1}, \text{ if } j = i-1 \\ -d_{k,i}d_{k,k-1}, \text{ if } j = k-1 \\ 0, \text{ otherwise} \end{cases}$

2) $P^{p^j}(M_{k;s_1,\dots,s_l}L_k^{p-2}) = \begin{cases} M_{k;s_1,\dots,s_t+1,\dots,s_l}L_k^{p-2};\ j = s_t,\ s_{t+1} \neq s_t+1 \\ (p-2)M_{k;s_1,\dots,s_l}L_k^{p-2}d_{k,k-1};\ j = k-1,\ s_l \neq k-1 \\ -L_k^{p-2}(M_{k;s_1,\dots,s_l}d_{k,k-1} + \sum\limits_{s'_t \notin \{s_1,\dots,s_l\}} (-1)^t M_{k;s_1,\dots,s'_t,\dots,s_l}d_{k,s_t}); \\ j = k-1,\ s_l = k-1 \\ 0;\ j = s_t - 1 = s_{t-1},\ l-1 = s_t, k \end{cases}$

Lemma 1. $P^{p^t}(d_{k,i}^{p^l}) = \begin{cases} d_{k,i-1}^{p^l}, \text{ if } t = l+i-1 \\ -d_{k,i}^{p^l}d_{k,i-1}^{p^l}, \text{ if } t = l+k-1 \\ 0, \text{ otherwise} \end{cases}$.

Theorem 4. [3] *1) Let* $q > 0$. *If* $q = \sum\limits_{i}^{k-1} a_t p^{t+l}$ *such that* $p-1 \geq a_t \geq a_{t-1} > a_{i-1} = 0$. *Then*

$$P^q d_{k,0}^{p^l} = d_{k,0}^{p^l}(-1)^{a_{k-1}} \prod_i^{k-1} \binom{a_t}{a_{t-1}} d_{k,t}^{p^l(a_t-a_{t-1})}$$

Otherwise, $P^q d_{k,0}^{p^l} = 0$.

2) Let $q = \sum\limits_{s}^{k-1} a_t p^{t+l} > 0$ *such that* $p-1 \geq a_t \geq a_{t-1} \geq a_i \geq 0$ *and* $a_i + 1 \geq a_{i-1} \geq a_t \geq a_{t-1} \geq a_{s-1} = 0$. *Then*

$$P^q d_{k,i}^{p^l} = d_{k,i}^{p^l}(-1)^{a_{k-1}} \left(\prod_{i+1}^{k-1}\binom{a_t}{a_{t-1}}\right)\binom{a_i+1}{a_{i-1}}\left(\prod_{s}^{i-1}\binom{a_t}{a_{t-1}}\right)\prod_{s}^{k-1} d_{k,t}^{p^l(a_t-a_{t-1})}$$

Here $a_{s-1} = 0$. *Otherwise,* $P^q d_{k,0}^{p^l} = 0$.

Remark 1. *Please note that the case* $a_i = 0$ *and* $a_{i-1} = 1$ *is allowed in the Theorem above.*

We shall apply formulas above on a Dickson algebra monomial starting with the lower non-zero p-th power.

Definition 1. *Let* $n = (n_0, \dots, n_{k-1})$ *be a sequence of non-negative integers and* $d^n = \prod_i d_{k,i}^{n_i}$ *a monomial in the Dickson algebra. Let* $n_i = \sum\limits_{t=0}^{l(i)} a_{i,t}p^{n_{i,t}}$ *be the* n_i*'s p-adic expansion with* $\prod a_{i,t} \neq 0$. *a) Let* $M := \{m_0, m_1, \dots, m_{l(n)} \mid m_i < m_{i+1}\} = \{n_{0,t} + k - 1, n_{i,s} + i - 1 \mid 0 \leq t \leq l(0),\ 1 \leq i \leq k-1 \text{ and } 0 \leq s \leq l(i)\}$.
b) Let $I(m_j, n) := (i_1, \dots, i_r)$ *such that* $m_j = n_{0,t} + k - 1 = n_{i_r,s} + i_r - 1$ *and* $i_{I(m_j,n)} := \begin{cases} \max I(m_j,n), \text{ if } 0 \notin I(m_j,n) \\ k, \text{ if } 0 \in I(m_j,n) \end{cases}$.

c) Let $P^{\Gamma(m,l)}$ *stand for the Steenrod operation* $P^{p^{m-l+1}}P^{p^{m-l+2}}\dots P^{p^m}$. *Let us call* $P^{\Gamma(m,l)}$ *a* ***Steenrod-Milnor action*** *of type* (m,l).

Proposition 1. *a)* $P^{\Gamma(m_0,k)}d_{k,0}^{p^{n_0}} = -d_{k,0}^{2p^{n_0}}$. *Here* $m_0 = n_0 + k - 1$.

b) $\underbrace{P^{\Gamma(m_0,k)}\dots P^{\Gamma(m_0,k)}}_{p-1} = -d_{k,0}^{p^{n_0+1}}$. *Here* $m_0 = n_0 + k - 1$.

c) Let $d^n \in D_k$ and $m_0 \in M$, then
$$P^{p^{m_0}} d^n = \sum_{0<i_r \in I(m_0,d^n)} a_{i_r,0} d^n d_{k,i_r-1}^{p^{n_{i_r,0}}} d_{k,i_r}^{-p^{n_{i_r,0}}} + d^n d_{k,0}^{a_{0,0}p^{n_{0,0}}} P^{p^{m_0}} d_{k,0}^{a_{0,0}p^{n_{0,0}}}.$$

d) Let $d^n \in D_k$ and $m_0 \in M$, then
$$P^{\Gamma(m_0,i_{I(m_0,n)})} d^n = \begin{cases} a_{i_{I(m_0,n)},0} d^n d_{k,0}^{p^{n_{i_{I(m_0,n)},0}}} d_{k,i_{I(m_0,n)}}^{-p^{n_{i_{I(m_0,n)},0}}}, & \text{if } 0 \notin I(m_0,d^n) \\ -a_{0,0} d^n d_{k,0}^{p^{n_{0,0}}}, & \text{if } 0 \in I(m_0,d^n) \end{cases}.$$

e) Let $d^n \in D_k$ and $m_0 \in M$, then $\underbrace{P^{\Gamma(m_0,i_{I(m_0,n)})} \ldots P^{\Gamma(m_0,i_{I(m_0,n)})}}_{a_{i_{I(m_0,n)},0}} d^n =$
$$\left(a_{i_{I(m_0,n)},0}\right)! d^n d_{k,0}^{a_{i_{I(m_0,n)},0} p^{n_{i_{I(m_0,n)},0}}} d_{k,i_{I(m_0,n)}}^{-a_{i_{I(m_0,n)},0} p^{n_{i_{I(m_0,n)},0}}} \quad \text{or}$$
$$\underbrace{P^{\Gamma(m_0,k)} \ldots P^{\Gamma(m_0,k)}}_{p-1-a_{0,0}} d^n = (-1)^{p-1-a_{0,0}} \frac{(p-1)!}{(a_{0,0}-1)!} d^n d_{k,0}^{p^{n_{0,0}+1}} d_{k,0}^{-a_{0,0}p^{n_{0,0}}}.$$

Proof. a) By lemma 1 $P^{p^t} d_{k,0}^{p^l} = 0$, if $t \neq l+k-1$ and $P^{p^t} d_{k,i}^{p^l} = 0$, if $t \neq l+k-1$ or $l+i-1$. Now the statement follows using Cartan formula.

b) is an application of a).

c) Since $m_0 = \max M$, Theorem 3 and Cartan formula implies the statement.

d) Let $m_0 = n_0 + k - 1 = n_i + i - 1$ for $i > 0$. By lemma 1 $P^{p^{m_0-i}} P^{p^{m_0-i+1}} \ldots P^{p^{m_0}} d_{k,i}^{p^{n_i}} = P^{p^{m_0-i}} d_{k,0}^{p^{n_i}} = P^{p^{n_i-1}} d_{k,0}^{p^{n_i}} = 0$. Now the statement is an application of c).

e) This is a repeated application of d). Two main cases should be considered depending on $i_{I(m_0,n)}$. Moreover, the number of times the S-M operation has to be applied depends on $a_{i_{I(m_0,n)},0}$. We describe the first step in details. The next steps follow the same pattern. Let us compare d^n and $d^n d_{k,0}^{p^{n_{i_{I(m_0,n)},0}}} d_{k,i_{I(m_0,n)}}^{-p^{n_{i_{I(m_0,n)},0}}}$. Let M and M' be the corresponding sets defined in definition 1.
Let $i_{I(m_0,n)} > 0$, then $n_{i_{I(m_0,n)},0} + k - 1 > m_0$. If $a_{i_{I(m_0,n)},0} = 1$ and $I(m_0,n) = \{i_{I(m_0,n)}\}$, then $m_0' = \min\{m_1, n_{i_{I(m_0,n)},0} + k - 1\} > m_0$. Otherwise, $m_0' = m_0$.
Let $i_{i_{I(m_0,n)},0} = 0$ and $a_{0,0} < p-1$, then $m_0' = m_0$. Otherwise, $m_0' = m_0 + 1$. Now the statement follows. ∎

Let us comment on the statement of last Proposition. Let d^n be a monomial and $d^{n'}$ the resulting monomial as in the statement of e) above. If for each index $i_r \in I(m_0,n)$ a suitable Steenrod-Milnor operation is defined, then the smallest p-th component of exponents of $d_{k,i}$'s are reduced and that of $d_{k,0}$'s is increased respectively.

Corollary 1. *Let $d^n \in D_k$ and $i_{r_t} \in I(m_0,n) = \{i_{r_1},\ldots,i_{r_l}\}$. a) If $0 < i_{r_1}$, then*
$$\underbrace{P^{\Gamma(m_0,i_{r_1})}}_{a_{i_{r_1},0}} \underbrace{P^{\Gamma(m_0,i_{r_2})}}_{a_{i_{r_2},0}} \ldots \underbrace{P^{\Gamma(m_0,i_{r_l})}}_{a_{i_l,0}} d^n = \lambda d^n d_{k,0}^{\left(\sum_{0<i_r \in I(m_0,n)} a_{i_r,0} p^{n_{i_t,0}}\right)} \prod_{i_r \in I(m_0,n)} d_{k,i_r}^{-a_{i_r,0}p^{n_{i_r,0}}}.$$
b) If $0 = i_{r_1}$, then
$$\underbrace{P^{\Gamma(m_0,i_{r_2})}}_{a_{i_{r_2},0}} \ldots \underbrace{P^{\Gamma(m_0,i_{r_l})}}_{a_{i_{r_l},0}} \underbrace{P^{\Gamma(m_0,k)}}_{p-1-a_{0,0}} d^n = \lambda d^n d_{k,0}^{\left(p^{k_{0,0}+1} + \sum_{0<i_r \in I(m_0,n)} a_{i_r,0} p^{n_{i_t,0}}\right)} \prod_{i_r \in I(m_0,n)} d_{k,i_r}^{-a_{i_r,0}p^{n_{i_r,0}}}.$$
Here $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$.

Proof. This is an application of Proposition 1 e). ∎

Let d^n be a monomial and $d^{n'}$ the resulting monomial as in last corollary. Let M and M' be as in definition 1, then $m_0 < m'_0$.

Definition 2. *Let m be a positive integer, $I = (i_1, ..., i_l)$ a strictly increasing sequence of integers between 0 and $k-1$, and $J = (a_1, ..., a_l)$ a sequence of integers between 0 and $p-1$. We define $P^{\Gamma(m,I,J)}$ the following S-M operation:*
a) if $i_1 = 0$, $P^{\Gamma(m,I,J)} = \underbrace{P^{\Gamma(m,i_2)}}_{a_2} ... \underbrace{P^{\Gamma(m,i_l)}}_{a_l} \underbrace{P^{\Gamma(m,k)}}_{p-1-a_1}$
b) If $0 < i_1$, $P^{\Gamma(m,I,J)} = \underbrace{P^{\Gamma(m,i_1)}}_{a_1} ... \underbrace{P^{\Gamma(m,i_l)}}_{a_l}$.

Theorem 5. *There exists a sequence of Steenrod-Milnor operations P^Γ such that $P^\Gamma d^n = \lambda d_{n,0}^{p^{l(n)}}$. Here $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$.*

Proof. We shall describe an algorithm which constructs the required sequence. This algorithm depends heavily on last corollary.

Step 0. Let $P^\Gamma = P^0$.

Step 1. Given d^n define $I(m_0, n)$, $J(m_0, n) = (a_{i_1,0}, ..., a_{i_l,0})$ and $i_{r(I(m_0,n))}$ as in Definition 1 b). Define $P^\Gamma := P^{\Gamma(m_0,I,J)} P^\Gamma$.

Step 2. Define $d^n := \lambda d^n d_{k,0}^{(\sum_{0<i_r \in I(m_0,n)} a_{i_r,0} p^{n_{i_t,0}})} \prod_{i_r \in I(m_0,n)} d_{k,i_r}^{-a_{i_r,0} p^{n_{i_r,0}}}$ or

$\lambda d^n d_{k,0}^{(p^{k_{0,0}+1} + \sum_{0<i_r \in I(m_0,n)} a_{i_r,0} p^{n_{i_t,0}})} \prod_{i_r \in I(m_0,n)} d_{k,i_r}^{-a_{i_r,0} p^{n_{i_r,0}}}$ given by corollary above.

If $n_i > 0$ for some $i > 0$ or $n_0 \neq p^{l(n)}$ for some positive integer $l(n)$, then proceed to step 1. Otherwise, the required sequence is P^Γ. ∎

Lemma 2. *Let d^n and $d^{n'}$ be monomials and $\{M, I(m_0, n), J(m_0, n)\}$, $\{M', I(m'_0, n'), J(m'_0, n')\}$ their corresponding sequences.*
a) If $m_0 = m'_0$, $I(m_0, n) = I'(m_0, n')$, and $J(m_0, n) = J(m_0, n')$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = 0$.
b) If $m_0 = m'_0$, $I(m_0, n) = I'(m_0, n')$, and $\exists t_0 > 0$ such that $a_{i_{t_0},0} > a'_{i_{t_0},0}$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = P^{\Gamma(m_0,I,J)}(d^n)$.
c) If $m_0 = m'_0$, $I(m_0, n) = I'(m_0, n')$, and $0 < a_{0,0} < a'_{0,0}$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = P^{\Gamma(m_0,I,J)}(d^n)$.
d) If $m_0 = m'_0$ and either $0 \notin I(m_0, n) \cap I(m_0, n')$ or $a_{0,0} = a'_{0,0}$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = P^{\Gamma(m_0,I,J)}(d^n)$.
e) If $m_0 = m'_0$ and either $0 < a_{0,0} < a'_{0,0}$ or $0 = a'_{0,0} < a_{0,0}$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = P^{\Gamma(m_0,I,J)}(d^n)$.
f) If $m_0 < m'_0$, then $P^{\Gamma(m_0,I,J)}(d^n - d^{n'}) = P^{\Gamma(m_0,I,J)}(d^n)$.

P^Γ as in the last Theorem is a repeated S-M action. Applying lemma above we define an ordering in D_k using the corresponding action and call it a **Steenrod-Milnor action ordering** and write S-M ordering.

Definition 3. *Let d^n, $d^{n'} \in D_k$ and $n_i = \sum_{t=0}^{l(i)} a_{i,t} p^{n_{i,t}}$, $n_i = \sum_{t=0}^{l'(i)} a'_{i,t} p^{n'_{i,t}}$. Here $\prod_{i,t} a_{i,t} \prod_{i,t} a'_{i,t} \neq 0$. 1) i) If $m_0 < m'_0$, we call $d^n < d^{n'}$.*

ii) If $m_0 = m'_0$ and one of hypotheses of last lemma is applied, we call $d^n < d^{n'}$.

iii) If $m_0 = m_0'$ and none of hypotheses of last lemma is applied, then the ordering is defined according to monomials $P^{\Gamma(m_0,I,J)}d^n$ and $P^{\Gamma(m_0,I,J)}d^{n'}$. Here $J = (a_{i_1,0}, ..., a_{i_l,0})$.

Next we extend the previous ideas to exterior monomials.

Lemma 3. *1) Let $M_{k;s_1,...,s_l}L_k^{p-2}d^n$ be a monomial in $(E(x_1,...,x_k) \otimes P[y_1,...,y_k])^{GL_k}$ and $P^B := \beta \underbrace{P^{p^0}\beta} \ldots \underbrace{P^{p^{k-l-2}} ... P^{p^0}\beta} \underbrace{P^{p^{k-l-1}} ... P^{p^{s_1}}} \ldots \underbrace{P^{p^{k-2}} ... P^{p^{s_l}}}$. Then*

$$P^B M_{k;s_1,...,s_l}L_k^{p-2}d^n = (-1)^{(k-l-1)!}d_{k,0}d^n$$

If $s_l = k-1$, then the result follows applying
$P^B := \beta \underbrace{P^{p^0}\beta} \ldots \underbrace{P^{p^{k-l-2}} ... P^{p^0}\beta} \underbrace{P^{p^{k-l-1}} ... P^{p^{s_1}}} \ldots \underbrace{P^{p^{k-3}} ... P^{p^{s_l-1}}}$.

2) Let $M_{k;s_1,...,s_l}L_k^{p-2}d^n$ and $M_{k;s_1',...,s_{l'}'}L_k^{p-2}d^{n'}$ be monomials such that $s_{l-t} < s_{l'-t}'$ and t is minimal with this property, then $P^B M_{k;s_1',...,s_{l'}'}L_k^{p-2}d^{n'} = 0$. Here P^B is as in 1).

Proof. Let us recall that $P^{p^{s_l}}(M_{k;s_1,...,s_l}L_k^{p-2}) = M_{k;s_1,...,s_l-1,s_l+1}L_k^{p-2}$ for $s_l < k-1$ and $P^{p^{s_l}}d_{k;t}^{a_t p^{n_t}} \neq 0$ if and only if $n_t = s_l - t + 1$ for $0 \leq t \leq s_l + 1$. If $0 = s_l$, we apply the Bockstein operation β. Thus $P^{p^{k-2}} ... P^{p^{s_l}} M_{k;s_1,...,s_l}L_k^{p-2}d^n = \sum_0^{k-1-s_l} M_{k;s_1,...,s_{l-1},s_l+t_l}L_k^{p-2}f_{t_l}$. Here f_{t_l} is a polynomial in D_k.
Let $P^E = \underbrace{P^{p^{k-l-1}} ... P^{p^{s_1}}} \ldots \underbrace{P^{p^{k-2}} ... P^{p^{s_l}}}$. Iterating the last formula we obtain:

$$P^E M_{k;s_1,...,s_l}L_k^{p-2}d^n = \sum_{q=1}^{l} \sum_0^{s_{q+1}+t_{q+1}-s_q} M_{k;s_1+t_1,...,s_{l-1}+t_{l-1},s_l+t_l}L_k^{p-2}f_{t_1,...,t_l}$$

Here $s_{l+1} = 0$ and $t_{l+1} = k-1$.
Let us suppose that $s_1 + t_1 < k - l$. Let $P^\Delta = P^{p^{k-l-2}} ... P^{p^0}\beta$ and $A = M_{k;s_1+t_1,...,s_{l-1}+t_{l-1},s_l+t_l}L_k^{p-2}f_{t_1,...,t_l}$. There are $s_1+t_1-1 \leq k-l-2$ positions to be filled by powers of y's using Steenrod operations: $\beta \underbrace{P^{p^0}\beta} \ldots \underbrace{P^{p^{k-l-2}} ... P^{p^0}\beta}$.
Since there are $k-l$ β's in this sequence and only $s_1+t_1-1 \leq k-l-2$ positions, it is obvious that $P^\Delta A = 0$. Now suppose that $s_1 + t_1 = k - l$ and one operation P^{p^q} of P^Δ is not applied on A. Then it will be less positions than the number of remaining β's. In that case $\beta \underbrace{P^{p^0}\beta} \ldots \underbrace{P^{p^{k-l-2}} ... P^{p^0}\beta} M_{k;s_1+t_1,...,s_{l-1}+t_{l-1},s_l+t_l}L_k^{p-2}f_{t_1,...,t_l} = 0$.
The claim follows. ∎

Definition 4. *3) i) $M_{k;s_1,...,s_l}L_k^{p-2}d^n < M_{k;s_1,...,s_l}L_k^{p-2}d^{n'}$, if $d_{k,0}d^n < d_{k,0}d^{n'}$.*

ii) $M_{k;s_1,...,s_l}L_k^{p-2}d^n < M_{k;s_1',...,s_{l'}'}L_k^{p-2}d^{n'}$, if $s_t < s_t'$ and t is maximal with this property.

Remark 2. *Because of our definitions, the S-M action ordering is a total ordering.*

Corollary 2. *Let $M_{k;s_1,...,s_l}L_k^{p-2}d^n \in (E_k \otimes S_k)^{GL_k}$. There exists a sequence of S-M operations $P^{\Gamma(M_{k;s_1,...,s_l}L_k^{p-2}d^n)}$ such that*

$$P^{\Gamma(d_{k,0}d^n)}\beta P^{p^0}\beta \ldots \underbrace{P^{p^{k-l-2}} ... P^{p^0}\beta} \underbrace{P^{p^{k-l-1}} ... P^{p^{s_1}}} \ldots \underbrace{P^{p^{k-2}} ... P^{p^{s_l}}} M_{k;s_1,...,s_l}L_k^{p-2}d^n = \lambda d_{k,0}^{p^q}$$

and q is minimal with this property.
Here $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$.

Now we are ready to proceed to our main Theorem.

Theorem 6. *Let $f : (E_k \otimes S_k)^{GL_k} \to (E_k \otimes S_k)^{GL_k}$ be a Steenrod module map which preserves the degree such that $f(d_{k,k-1}) = \lambda d_{k,k-1}$ for $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$. Then f is a lower triangular map with respect to S-M ordering and hence an isomorphism.*

Proof. By hypothesis and Theorem 3, $f(d_{k,i}) = \lambda d_{k,i}$ for $i = 0, ..., k-1$ after applying a suitable Steenrod operation.

Let $d^n \in D_k$ and $(d^{n(1)}, ..., d^{n(l(|d^n|))})$ the increasing sequence of elements of degree $|d^n|$. Let $f(d^n) = \sum_{t=1}^{l|d^n|} a_t d^{n(t)}$. Claim: If $d^{n(t_0)} = d^n$, then $a_t \equiv 0 \bmod p$ for $t < t_0$. We use induction on t for $t < t_0$. $P^{\Gamma(d^{n(1)})} f(d^n) = P^{\Gamma(d^{n(1)})} \sum_{t=1}^{l|d^n|} a_t d^{n(t)}$ implies $a_1 \equiv 0 \bmod p$. $P^{\Gamma(d^{n(i)})} f(d^n) = P^{\Gamma(d^{n(i)})} \sum_{t=i}^{l|d^n|} a_t d^{n(t)}$ implies $a_i \equiv 0 \bmod p$ for $i < t_0$. Now using Proposition 1 and the fact $f(d_{k,0}) = \lambda d_{k,0}$ for $\lambda \not\equiv 0 \bmod p$, we conclude that $a_{t_0} \not\equiv 0 \bmod p$. Hence f is a lower triangular map.

Because of the direct sum decomposition of the ring of invariance, it follows that $f(M_{k;s_1,...,s_l} L_k^{p-2} d^n) = \alpha M_{k;s_1,...,s_l} L_k^{p-2} d$, then
$\beta P^{p^0} \beta ... \underbrace{P^{p^{k-l-2}} ... P^{p^0}} \beta \underbrace{P^{p^{k-l-1}} ... P^{p^{s_1}}} ... \underbrace{P^{p^{k-2}} ... P^{p^{s_l}}} f(M_{k;s_1,...,s_l} L_k^{p-2} d^n) = \lambda d_{k,0} d$
and the claim follows. ∎

Remark 3. *Please note that for $k = 1$ it suffices to require $f(M_{1;1} L_1^{p-2}) = \lambda(M_{1;1} L_1^{p-2})$, since $\beta(M_{1;1} L_1^{p-2}) = d_{1,0}$.*

Corollary 3. *a) Let $S(E_k \otimes S_k)^{GL_k}$ be the subalgebra of $(E_k \otimes S_k)^{GL_k}$ generated by $\{d_{k,i}, M_{k,s_1,...,s_{k-1}}, M_{k,s'_1,...,s'_{k-3},k-1}\}$ where $0 \le i \le k-1$, $0 \le s_1 < ... < s_{k-1} \le k-1$ and $0 \le s'_1 < ... < s'_{k-3} \le k-2$. If $f : S(E_k \otimes S_k)^{GL_k} \to S(E_k \otimes S_k)^{GL_k}$ satisfies $f(d_{k,k-1}) = \lambda d_{k,k-1}$, then f is an isomorphism.*

b) Let $I[k]$ be the ideal of $S(E_k \otimes S_k)^{GL_k}$ generated by $\{d_{k,0}, M_{k,s_1,...,s_{k-1}}, M_{k,s'_1,...,s'_{k-3},k-1}\}$, then the induced map f which satisfies $f(d_{k,0}) = \lambda d_{k,0}$ is also an isomorphism.

Corollary b) above is a reformulation of Theorem 4.1 in [1]. We close this work by applying last corollary in the mod $-p$ homology of QS^0.

Let $R =< Q^{(I,J)} | I = (i_1, ..., i_n), J = (\varepsilon_1, ..., \varepsilon_n) >$ be the Dyer-Lashof algebra, then $H_*(Q_0 S^0; \mathbb{Z}/p\mathbb{Z})$ is the free commutative algebra generated by $\Phi(R)$ subject to the following relation $Q^{(I,J)} \approx (Q^{(I',J')})^p$ if $I = (i_1, I')$, $J = (0, J')$ and $exc(Q^{(I,J)}) = 0$. Here $\Phi : R \to H_*(Q_0 S^0; \mathbb{Z}/p\mathbb{Z})$ is the A_*-module map given by $\Phi(Q^{(I,J)}) = Q^{(I,J)}[1] * [-p^{l(I)}]$, $[1]$ is a generator of $\tilde{H}_0(S^0; \mathbb{Z}/p\mathbb{Z})$, $[r] = [1]^r$ and $l(I)$ is the length of I. Thus there exists an A_*-module isomorphism between the generators of $H_*(Q_0 S^0; \mathbb{Z}/p\mathbb{Z})$ and the quotient $R/Q_0 R$ where $Q_0 R = \{Q^{(I,J)} | exc(I, J) = 0\}$. It is known that $R[k]^* \cong S(E_k \otimes S_k)^{GL_k}$ as Steenrod algebras and $(R/Q_0 R)[k]^* \cong I[k]$ as Steenrod modules. Here $R = \bigoplus R[k]$. Now the following Theorem is a consequence of last corollary.

Theorem 7. [1]*Let $f : \Omega_0^\infty S^\infty \to \Omega_0^\infty S^\infty$ be an H-map which induces an isomorphism on $H_{2p-3}(\Omega_0^\infty S^\infty; \mathbb{Z}/p\mathbb{Z})$. If $p > 2$ suppose in addition that f is a loop map* *or that*

$$f_*(d_{2,0})^* = \lambda(d_{2,0})^*$$

for some $\lambda \in (\mathbb{Z}/p\mathbb{Z})^*$. *Then* $f_{(p)}$ *is a homotopy equivalence. Here* $(d_{2,0})^*$ *is the* hom-*dual of the top degree Dickson generator in* D_2.

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